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Expansive Flows and the Fundamental Group

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Abstract. In this paper we construct stable and unstable foliations for expansive flows operating on 3-manifolds. We also prove that the fundamental group of the manifold has exponential growth.

I. Introduction

In ([**7**]) Lewowicz proved (see also [**3**]) that an expansive homeomorphism of a surface is conjugate to an Anosov or Pseudo Anosov map. This result and the methods used to prove it indicate that several facts of hyperbolic dynamics hold in the expansive case.

A flow ϕ_t acting on a metric space K without fixed points is said to be expansive if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathrm{dist}(\phi_t(x), \phi_{\sigma(t)}(y)) < \delta$ for every $t \in \mathbb{R}$, some x, $y \in K$ and some homeomorphism $\sigma\colon \mathbb{R} \to \mathbb{R}$ with $\sigma(0) = 0$, then $y = \phi_T(x)$, where $|\tau| < \varepsilon$.

Anosov flows and suspensions of Pseudo Anosov maps are examples of expansive flows.

Here we study expansive flows and we obtain some results not depending on the dimension of the manifold, namely, the existence of a Lyapunov function and the non-existence of stable points. However, we construct stable and unstable foliations only in the 3-dimensional case due to the nature of the techniques of Lewowicz.

Let $\pi_1(M)$ be the fundamental group of M. We say that a finitely generated group has exponential growth if given a finite set of generators the function $\Gamma(n) =$ (number of distinct group elements of word-length $\leq n$) dominates $A \exp(an)$, for some $A > 0$, $a > 0$. This definition is

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equivalent to the function $P(r)$ = (number of distinct free homotopy classes of loops in M of length $\leq r$) dominating $B\exp(br)$, for some $B > 0$, $b > 0$, in the case the group is the fundamental group of a manifold M (see [**8**]).

We prove the following

Theorem. *Let M be a compact connected 3-manifold and $\phi_t\colon M \to M$ an expansive flow, then $\pi_1(M)$ has exponential growth.*

A similar result was proved by Plante and Thurston ([**8**]) for codimension one Anosov flows and by Margulis ([**1**]) for Anosov flows on 3-manifolds.

We show here that a proof similar to that of [**8**] also works. The exponential growth is obtained in the following way (see section VI of this paper): A piece of unstable manifold, whose length is bounded from below, flows forward to give exponentially many such pieces of unstable manifold connected by pieces of trajectory (in the absence of hyperbolicity this construction needs some care). The resulting long path gives rise to exponentially many loops based in a single stable disc, which are pairwise non-homotopic.

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II. Lyapunov functions

Let M be a compact connected 3-manifold endowed with a Riemannian structure and $\phi\colon M \times \mathbb{R} \to M$ be a flow without fixed points.

In the following paragraph we obtain a continuous family of local sections topologically transversal to the flow; i.e. a continuous function $x \mapsto H(x)$, where $H(x)$ is a local section through x which is topologically transversal to the flow (see [**2**], IV, 1.3 and the references therein). If the flow is C^1 it is enough to define each section by

$$H(x) = \{\exp_x v \text{ such that } |v| < \varepsilon, \langle X(x), v\rangle = 0\}$$

where $X(x) = \phi(x, 0)$ and ε is a suitable fixed positive number.

Assume that our flow is only continuous. Let $B_\varepsilon(x)$ stand for the open ball of centre 0 and radius ε. Define $B(\varepsilon)$ by

$$B(\varepsilon) = \{(x, y) \in M \times M \text{ such that } \operatorname{dist}(x, y) < \varepsilon\}$$

Take a finite open covering of M by sets G_i defined as

$$G_i = \{(\phi(x, t) \mid x \in D_i, |t| < \varepsilon \text{ and } D_i \text{ is a local section}\}$$

Set $D_i(x) = \phi_t(D_i)$, where t is the unique real number such that $\phi_{-t}(x) \in D_i$ and $|t| < \varepsilon$.

Let r be the Lebesgue number of the covering $\{G_i\}$. Define continuous functions $\sigma_i: G_i \times G_i \to \mathbb{R}$ by the condition

$$\phi(y, \sigma_i(x, y)) \in D_i(x)$$

Let $\{f_i\}$ be a partition of unity subordinate to the covering $\{G_i \times G_i\}$ (of the set $B(r)$). Next define $\sigma: B(r) \to \mathbb{R}$ by

$$\sigma = \sum_i f_i \sigma_i$$

Define a local section $H(x)$ as

$$H(x) = \{\phi(y, \sigma(x, y)) \mid (x, y) \in B(r)\}$$

Let $H_\varepsilon(x)$ be the connected component of $B_\varepsilon(x) \cap H(x)$ that contains x. Define $N(\varepsilon)$ by

$$N(\varepsilon) = \{(x, y) \mid y \in H_\varepsilon(x)\}$$

Take β such that $\operatorname{dist}(\phi_t(x), x) < r/3$ for $|t| \leq \beta$. If $\operatorname{dist}(x, y) < r/3$ and $|t| \leq \beta$, then $\sigma \stackrel{\text{def}}{=} \sigma(\phi_t(x), y)$ is the unique real number close to 0 for which

$$\phi(y, \sigma) \in H_r(\phi_t(x))$$

Define a continuous function $\hat{\tau}: B(r/3) \times [-\beta, \beta] \to \mathbb{R}$ by

$$\hat{\tau}(x, y, t) = \sigma(\phi_t(x), y)$$

Remark. It is easy to check that ϕ is expansive if and only if there is a number $0 < \alpha_0 < r/3$, such that if for a pair of points x, y there exists a continuous, surjective, increasing function $\tau_{x,y}\colon \mathbb{R} \to \mathbb{R}$, $\tau_{x,y}(0) = 0$, with the property

$$\phi(y, \tau_{x,y}(t)) \in H_{\alpha_0}(\phi_t(x))$$

$\forall t \in \mathbb{R}$, then $x = y$.

Note. We use the notation $\tau_{x,y}(t)$ and $\tau(x, y, t)$ interchangeably and similarly with $\phi_t(x)$ and $\phi(x, t)$. In the sequel $\overline{A}$ and $\operatorname{clos} A$ will stand for the closure of A.

Remark. Let δ be a positive number, $\delta < \alpha_0/3$.

Suppose that for fixed $t > 0$ there exists a continuous increasing function $\tau_{x,y}\colon [0, t] \to \mathbb{R}$ such that $\phi(y, \tau(x, y, x)) \in H_{3\delta}(\phi_s(x))$ for $s \in [0, t]$.

Choose $0 = t_0 < t_1 < \ldots < t_p = t$ so that $t_{i+1} - t_i < \beta/2$, then we have:

$$\tau(x, y, s) = \tau(x, y, t_i) + \widehat{\tau}(\phi(x, t_i), \phi(y, \tau(x, y, t_i)), s - t_i)$$

for $s \in (t_i, t_{i+1}]$.

This allows us to express τ in terms of $\widehat{\tau}$, which is continuous in x, y. On account of this, for $(\bar{x}, \bar{y})$ close to (x, y), we can find a continuous increasing function $\tau_{\bar{x},\bar{y}}\colon [0, t] \to \mathbb{R}$ such that $\phi(\bar{y}, \tau_{\bar{x},\bar{y}}(s)) \in H_{3\delta}(\phi_s(\bar{x}))$ for $s \in [0, t]$.

Take δ, $0 < \delta < \alpha_0/3$ and define C^- as

$$C^- = \left\{ \begin{array}{l} (x, y) \in M \times M \ \text{ such that there exists a continuous,} \\ \text{increasing, surjective function } \tau_{x,y}\colon \mathbb{R}^- \to \mathbb{R}^-, \tau_{x,y}(0) = 0, \\ \text{with the property } \phi(y, \tau_{x,y}(t)) \in \operatorname{clos} H_{2\delta}(\phi_t(x)), \forall t \in \mathbb{R}^- \end{array} \right\}$$

The last remark easily implies that C^- is compact.

Several of the following ideas are from [**5**].

On account of the compactness of C^- there exists $t^* > 0$ such that if $(x, y) \in C^-$ and $\operatorname{dist}(x, y) = \delta$, then there is some continuous increasing function $\tau_{x,y}\colon [0, t_1(x, y)] \to \mathbb{R}$ such that $\operatorname{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) > \delta$ for some t with $0 < t \leq t_1(x, y) \leq t^*$.

Choose $\rho > 0$ such that

$$\min\{\operatorname{dist}(\phi(x, t), \phi(y, \tau(x, y, t)) \ \mid \ 0 \leq t \leq t_1(x, y)\} > \rho.$$

We claim that there exist σ, $0 < \sigma < \rho$ and δ', $0 < \delta' < \delta$ such that if $(x, y) \in N(\delta), \mathrm{dist}(x, y) \le \sigma$ and there exists t_0 so that

$$\mathrm{dist}(\phi(x, t_0), \phi(y, \tau(x, y, t_0))) > \delta$$

and

$$\mathrm{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) \le \delta$$

for $0 \le t \le t_0$, then

$$\mathrm{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) > \delta$$

for some t with $t_0 \le t \le t_0 + t^*$ and also

$$\mathrm{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) > \delta$$

for $t_0 \le t \le t_0 + t_1(x, y)$.

If this were not true, there would exist sequences, $(x_n, y_n) \in N(\delta)$, $\mathrm{dist}(x_n, y_n) \to 0$, $\delta'_n \to \delta$ and $t_n \to \infty$ so that

$$\mathrm{dist}(\phi(x_n, t), \phi(y_n, \tau(x_n, y_n, t))) \le \delta$$

for $0 \le t \le t_n + t^*$ and

$$\delta'_n \le \mathrm{dist}(\phi(x_n, t_n), \phi(y_n, \tau(x_n, y_n, t_n))) \le \delta.$$

Set $z_n = \phi(x_n, t_n)$, $w_n = \phi(y_n, \tau(x_n, y_n, t_n)$ and let x, y be limit points of $\{z_n\}$ and $\{w_n\}$ respectively. For $t \ge 0$,

$$\lim_n \mathrm{dist}(\phi(z_n, t), \phi(w_n, \tau(z_n, w_n, t))) = \\ = \mathrm{dist}(\phi(x, t), \phi(y, \tau(x, y, t)));$$

then $(x, y) \in C^-$ and $\mathrm{dist}(x, y) = \delta$.

For $0 \le t \le t^*$, we have

$$\mathrm{dist}(\phi(x_n, t + t_n), \phi(y_n, \tau(x_n, y_n, t + t_n))) = \\ = \mathrm{dist}(\phi(z_n, t), \phi(w_n, \tau(z_n, w_n, t))) \le \delta.$$

This implies $\mathrm{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) \le \delta$, which contradicts the choice of t^*.

An analogous argument allows us to choose σ and δ' so that the second claim holds.

Set $A \overset{\text{def}}{=} \{(x,y) \mid (x,y) \in \operatorname{clos} N(2\delta) \text{ and } \operatorname{dist}(x,y) \geq \delta\}$. Let $a: M \times M \to \mathbb{R}$ be a smooth function

$$\begin{aligned} a(x,y) &= 1 && \text{if } (x,y) \in N(2\delta), \operatorname{dist}(x,y) \leq \sigma \\ a(x,y) &= 0 && \text{if } (x,y) \in A \\ a(x,y) &> 0 && \text{elsewhere} \end{aligned}$$

Suppose first that our flow is C^1. Then we may assume that $\widehat{\tau}(x,y,0)$ is defined in $M \times M$ (extending it arbitrarily).

Let Y be the following vector field on $M \times M$

$$Y(x,y) = a(x,y)(X(x), \widehat{\tau}(x,y,0)X(y)).$$

Let $\psi((x,y),t)$ be the flow of Y and set $\operatorname{dist}\psi((x,y),t) = \operatorname{dist}(z,w)$ for $(z,w) = \psi((x,y),t)$. Observe that for $(x,y) \in N(\sigma)$ and small t we have $\psi((x,y),t) = (\phi(x,t), \phi(y, \widehat{\tau}(x,y,t)))$.

Let $d: M \times M \to \mathbb{R}$ be a smooth non-negative function that vanishes on

$$C^- \bigcup \{(x,y) \mid (x,y) \in \operatorname{clos} N(2\delta) \text{ and } \operatorname{dist}(x,y) \geq \rho\}$$

and only there.

Remark. Note that $\lim_{t\to\infty} d(\psi((x,y),t)) = 0$ if $(x,y) \in N(\sigma)$ because either $\operatorname{dist}\psi((x,y),T) \geq \delta'$ for some $T > 0$ and therefore $d(\psi((x,y),t)) = 0$ if $t > T$ or $\operatorname{dist}\psi((x,y),t) \leq \delta'$ for $t \geq 0$ which implies

$$\operatorname{dist}(\phi(x,t), \phi(y, \tau(x,y,t))) \leq \delta'$$

for $t \geq 0$ and consequently $\psi((x,y),t) \to C^-$ if $t \to \infty$.

Assume now that ϕ is only continuous and set

$$\Phi((x,y),t) \overset{\text{def}}{=} (\phi(x,t), \phi(y, \tau(x,y,t))).$$

As in [**7**] define $f(t) = \sup\{a(\Phi((x,y),t)) \mid (x,y) \in A\}$. Then f is continuous and $\lim_{t\to 0} f(t) = 0$. The same arguments of lemma 1.1 of [**7**] give the existence of a continuous increasing function $h: \mathbb{R} \to \mathbb{R}$ such that for any λ

$$\int_0^\lambda [h(f(t))]^{-1} dt = \infty.$$

Moreover, such an h may be taken smooth, $h'(t) > 0$ if $t \neq 0$, and $h(1) = 1$. Put $g = h \circ a$; then g vanishes on A, is positive on $N(2\delta) - A$, $g = 1$ if $a = 1$, and for any λ

$$\int_0^\lambda [g(\Phi((x,y),t))]^{-1} dt = \infty$$

if $(x,y) \in A$.

Let us define ψ. For $(x,y) \in N(2\delta) - A$ set

$$\psi((x,y),t) = \Phi((x,y),\sigma(t))$$

where $\sigma(t) = \sigma_{x,y}(t)$ is the inverse function of

$$t = \int_0^\sigma [g(\Phi((x,y),t))]^{-1} ds.$$

Put $\psi((x,y),t) = (x,y)$ if $(x,y) \in A$.

The arguments of lemma 1.2 of [**7**] show that our last remark holds for this new ψ.

Now we can prove as in [**5**].

Lemma 1. *Given $\varepsilon > 0$, there exists $T > 0$ such that $d(\psi((x,y),t)) < \varepsilon$, for $t \geq T$ and $(x,y) \in N(\sigma)$.*

Proof. Because of the way δ, δ', ρ were chosen, if for some $\varepsilon_0 > 0$ we had pairs $(x,y) \in N(\sigma)$ so that $d(\psi((x,y),t)) \geq \varepsilon_0$ for arbitrarily large t, we would have pairs $(x_n, y_n) \in N(\sigma)$ and continuous increasing functions $\tau_{x_n,y_n}: [0,t_n] \to \mathbb{R}$, with $t_n \to \infty$ such that

$$\operatorname{dist}(\phi(x_n,t), \phi(y_n, \tau(x_n,y_n,t))) \leq \delta$$

for $0 \leq t \leq t_n$ and

$$d(\phi(x_n,t_n), \phi(y_n, \tau(x_n,y_n,t_n))) \geq \varepsilon_0$$

Then (x_∞, y_∞), a limit point of $(\phi(x_n,t_n), \phi(y_n, \tau(x_n,y_n,t_n))$, would be in C^- and $d(x_\infty, y_\infty) \geq \varepsilon_0$, a contradiction. □

For $U: N(\sigma) \to \mathbb{R}$, define $\dot{U}: N(\sigma) \to \mathbb{R}$ (the derivative of U along ϕ) as

$$\dot{U}(x,y) = \frac{d}{dt}\bigg|_{t=0} U(\phi(x,t), \phi(y, \hat{\tau}(x,y,t)))$$

Let $\ddot{U}$ denote the derivative of $\dot{U}$. A function U as above is said to be positive definite if $U(x,y) \geq 0$ and $U(x,y) = 0$ if and only if $x = y$.

Lemma 2. *There exists a smooth function $e: \mathbb{R}^+ \to \mathbb{R}^+$, $e(0) = 0$, $e'(s) > 0$ if $s \neq 0$, with the property that the function $V: N(\sigma) \to \mathbb{R}$ defined by*

$$V(x,y) = \int_0^\infty e[d(\psi((x,y),t))]dt$$

has derivative $\dot{V}$ given by

$$\dot{V}(x,y) = -e(d(x,y))$$

Proof. There exists a function e, in the above conditions which makes the integral uniformly convergent (see [**5**]). For small u

$$\begin{aligned} V(\phi(x,u), \phi(y, \hat{\tau}(x,y,u))) &= V(\psi((x,y),u)) \\ &= \int_0^\infty e\{d[\psi((x,y), t+u)]\}dt \\ &= \int_u^\infty e\{d[\psi((x,y),t)]\}dt, \end{aligned}$$

and then $\dot{V}(x,y) = -e(d(x,y))$. □

Remark. Define the function $\widehat{V}: B(\sigma) \to \mathbb{R}$ by

$$\widehat{V}(x,y) = \int_0^\infty e\{d[\psi((x, P_x(y)),t)]\}dt$$

where $P_x(y) = \phi(y, \hat{\tau}(x,y,0))$. Then $\widehat{V}$ is a continuous extension of V.

On account of the previous remark the arguments of [**5**] (p. 202) hold and consequently we obtain

Lemma 3. *There exist $\alpha > 0$ and a function $U: N(\alpha) \to \mathbb{R}$, such that both U and $\ddot{U}$ are positive definite and admit continuous extensions to $B(\alpha)$.*

III. Stable points

In this part we follow the ideas of [**6**].

Definition. A point x of M is said to be a stable point of ϕ if given $\varepsilon > 0$ there exists $\delta > 0$ so that if $y \in H_\delta(x)$, then there exists a continuous function $\tau_{x,y}: \mathbb{R}^+ \to \mathbb{R}^+$, $\tau_{x,y}(0) = 0$ such that $\phi(y, \tau(x, y, t)) \in H_\varepsilon(\phi_t(x))$, for $t > 0$.

Let us define $K_k(x, t)$ as

$$K_k(x,t) = \{y \mid y \in H_\alpha(\phi_t(x)) \text{ and } U(\phi_t(x), y) \leq k\},$$

where α and U are as in lemma 3.

Lemma 4. *An expansive flow has no stable points.*

Proof. Let x be a stable point of ϕ and let ε and δ be as in the definition of stable point, with $\delta < \varepsilon < \alpha$.

Choose $k > 0$ such that if $y \in K_k(x,t)$ then $\mathrm{dist}(\phi_t(x), y) < \delta$. Therefore if $y \in \partial K_k(x, 0)$, then $\phi(y, \tau(x, y, t)) \in H_\varepsilon(\phi_t(x))$ for $t \geq 0$.

We shall see that the expansivity of ϕ implies that there exists $T < 0$ so that $U(\phi_t(x), y) < 0$ for $y \in \partial K_k(x,t)$ and $t \leq T$.

If this were not true, for $T < 0$ arbitrarily large there would exist an arc $a: [0,1] \to K_k(x, T)$ such that $a(0) = \phi_T(x)$, $\dot{U}(\phi_T(x), a(1)) \geq 0$ and $a(1) \in \partial K_k(x, T)$.

Let s_0 be the supremum of those $s \in [0,1]$ so that

$$\phi(a(s), \tau(\phi_T(x), a(s), t)) \in K_k(x,t),$$

for $t \in [0, -T]$. Then, $s_0 < 1$. Let

$$Y_T = \phi(a(s_0), \tau(\phi_T(x), a(s_0), -T)),$$

then $Y_T \in \partial K_k(x, 0)$ and $\phi(Y_T, \tau(x, y_T, s)) \in K_k(x, s)$ for $s \in [T, 0]$.

This implies that the entire trajectory of a limit point of Y_T is ε-close to the trajectory of x, which is absurd.

Take $z \in \alpha(x)$ (the α-limit set of x). Then we have $\dot{U}(\phi_t(z), y) \leq 0$, with $y \in \partial K_k(z,t)$ and $t \in \mathbb{R}$. Therefore if $y \in K_k(z, t_0)$, for some $t_0 \in \mathbb{R}$, we have $\phi(y, \tau(z, y, t)) \in K_k(z,t)$ for $t \geq t_0$ and then $\phi_t(z) \in K_k(x, 0)$ for arbitrarily large values of t.

As k was chosen arbitrarily, $x \in \omega(z)$ (the ω-limit set of z). This implies $x \in \omega(x)$, because the trajectories of x and z are asymptotic in the future.

Using the same argument for $y \in H_r(x)$, with an appropriate $r > 0$, we see that $y \in \omega(y) = \omega(x)$.

As a consequence of this, there exists an open set B such that $H_r(x) \subset B \subset \omega(x)$.

If $y' \in \omega(x)$, given k there exists t, so that $y' \in K_k(\phi_t(z), 0)$ and then $\phi(y', \tau(\phi_t(z), y', u)) \in K_k(\phi_t(z), u)$ for $u \geq 0$. This means that there exists $T > 0$ such that $\phi_T(y') \in H_r(x)$ and then $y' \in \phi_{-T}(H_r(x)) \subset \phi_{-T}(B) \subset \omega(x)$, i.e., $\omega(x)$ is open.

As M is connected $\omega(x) = M$, which implies that every point of M is stable and therefore $\omega(y) = M$, $\forall y \in M$.

Now choose positive numbers δ, k_1, k_2 such that $H_\delta(x) \subset K_{k_1}(x, 0)$, $K_{k_2}(x, 0) \subset H_{\delta/2}(x)$ for every $x \in M$ and $\dot{U}(\phi_t(z), y) \leq 0$ for every $t \in \mathbb{R}$ and $y \in K_{k_i}(z, t)$, $i = 1, 2$.

A compactness argument on $\operatorname{clos} H_\delta(z)$ allows us to find $T > 0$ so that $\phi(y, \tau(z, y, t)) \in H_{\delta/2}(\phi_t(z))$ for $t \geq T$ and $y \in \operatorname{clos} H_\delta(z)$.

Let $h > 0$ be such that if $\operatorname{dist}(x, y) < h$, the Poincaré map

$$P_y \colon H_{\delta/2}(x) \to H_\delta(y)$$

is defined. Therefore, if we choose $t \geq T$ so that $\operatorname{dist}(\phi_t(z), z) < h$, a suitable return map

$$P \colon \operatorname{clos} H_\delta(z) \to \operatorname{clos} H_\delta(z)$$

is defined and then ϕ would have a periodic orbit, which contradicts $\omega(y) = M$, $\forall y \in M$. □

IV. Local product structure

Let us choose numbers $0 < \delta_1 < \delta_2 < \alpha$ and $k > 0$ such that $H_{\delta_1}(x) \subset \{y \mid U(x, y) \leq k\} \subset H_{\delta_2}(x)$ and prove as in [**7**].

Lemma 5. *Let A be an open set of $H_{\delta_2}(x)$, $x \in A \subset H_{\delta_1}(x)$. Then there exists a compact connected set C, $x \in C \subset \overline{A}$, $C \cap \partial A \neq \phi$ such that for every $y \in C$ there exists a continuous increasing surjective function $\tau_{x,y} \colon \mathbb{R}^+ \to \mathbb{R}^+$, with $\tau_{x,y}(0) = 0$ and $\phi(y, \tau(x, y, t)) \in H_{\delta_2}(\phi_t(x))$, for $t \geq 0$.*

Proof. Suppose this is not true. Then there exists $T_0 > 0$ such that for every compact, connected set $D \subset \overline{A}$ joining x to ∂A, there exists $z \in D$,

$0 \le t \le T_0$, so that $\phi(z, \tau(x, z, t)) \notin H_{\delta_2}(\phi_t(x))$ for an appropriate τ. For otherwise we could find numbers $T_n \to \infty$ and compact connected sets $D_n \subset \overline{A}$, joining x to ∂A, such that for every $y \in D_n$ and appropriate τ, $\phi(y, \tau(x, y, t)) \in H_{\delta_2}(\phi_t(x))$ for $0 \le t \le T_n$. But then the set

$$D_\infty = \bigcap_n^\infty \operatorname{clos}(\bigcap_{n=j}^\infty D_j)$$

will satisfy the thesis of the lemma; a contradiction.

For arbitrarily large T, the boundary of the connected component of $K_k(x, T)$ that contains $\phi_T(x)$ must have points y such that $\dot{U}(\phi_T(x), y) < 0$ because if this were not true, taking limits as $t \to \infty$ we would get that any $y \in \omega(x)$ would be a stable point of ϕ_{-t}; which is absurd by lemma 4.

Choose $T > T_0$. Take an arc $a: [0, 1] \to K_k(x, T)$, $a(0) = \phi_T(x)$, $a(1) \in \partial K_k(x, T)$ and $\dot{U}(\phi_T(x), a(1)) < 0$. Let s_0 be the supremum of those $s \in [0, 1]$ such that $\phi(a(s), \tau(\phi_T(x), a(s), t)) \in K_k(\phi_T(x), t)$ for $-T \le t \le 0$ and $\phi(a(s), \tau(\phi_T(x), a(s), -T)) \in A$. Then $s_0 < 1$, and since $\ddot{U} > 0$ we have that $\phi(a(s_0), \tau(\phi_T(x), a(s_0), t)) \in K_k(\phi_T(x), t)$ for $-T \le t \le 0$. Thus $\phi(a(s_0), \tau(\phi_T(x), a(s_0), -T)) \in \partial A$ and the set

$$D = \{\phi(a(s), \tau(\phi_T(x), a(s), -T)) \mid 0 \le s \le s_0\}$$

has the property that if $y \in D$, then $\phi(y, \tau(x, y, t)) \in H_{\delta_2}(\phi_t(x))$ for $0 \le t \le T$; a contradiction. □

Remark. Recall that lemma 3 permit to find a number $\alpha > 0$ such that U and its derivatives are defined on $N(\alpha)$. Observe also that the function $\widehat{\tau}$, defined at the beginning of section II, is defined on $B(\alpha) \times [-\beta, \beta]$ and consequently if $y \in H_\alpha(x)$, the Poincaré map

$$P_x: H_\alpha(y) \to H_{\alpha_0}(x)$$

is well defined.

Definition. Let $\delta < \alpha$. Define $S_\delta(x)$ as $S_\delta(x) = \{y \in H_\delta(x) |$ there exists a continuous increasing and surjective function $\tau_{x,y}: \mathbb{R}^+ \to \mathbb{R}^+$, $\tau_{x,y}(0) = 0$, with the property $\phi(y, \tau(x, y, t)) \in H_\delta(\phi_t(x))$ for $t \ge 0\}$. $U_\delta(x)$ is defined analogously for $t \le 0$.

For $y \in H_\alpha(x)$ and $\delta < \alpha$ we define the δ-stable set of y in $H_\alpha(x)$ as

$$S_\delta^x(y) = P_x(S_\delta(y))$$

and the δ-unstable set of y in $H_\alpha(x)$ as

$$U_\delta^x(y) = P_x(U_\delta(y)).$$

Lemma 6. *Given $0 < \delta < \alpha$, there is $\sigma_0 > 0$ such that if $0 < \sigma < \sigma_0$, then*

(1) *$P_y(x) \in S_\delta(y)$ if $y \in S_\sigma(x)$*

(2) *$y \in S_\delta^x(z)$ if $z \in S_\sigma^x(y)$*

(3) *$q \in S_\delta^x(p)$ if $p, q \in S_\sigma^x(y)$*

(4) *if $y \in H_\sigma(x)$ there exists a compact connected set contained in $S_\delta^x(y) \cap H_\delta(x)$, joining y to $\partial H_\sigma(x)$.*

(5) *$S_\sigma^x(y)$ does not separate $H_\alpha(x)$.*

Proof. (1) For $t > 0$, define

$$\widetilde{\tau}_{y,x}(t) = -\widehat{\tau}(y, x, 0) + \tau_{x,y}^{-1}(t) + \widehat{\tau}(\phi_t(y), \phi(x, \tau_{x,y}^{-1}(t)), 0)$$

which is continuous, surjective, increasing and $\widetilde{\tau}_{y,x}(0) = 0$. Set $z = P_y(x)$, if σ is small enough we obtain $z \in H_\delta(y)$ and hence $\phi(z, \widetilde{\tau}_{y,x}(t)) \in H_\delta(\phi_t(y))$ for $t \geq 0$.

(2) Set $\bar{z} = P_y(z)$ and take σ small such that $\phi(y, \tau(\bar{z}, P_{\bar{z}}(y), t) \in H_\delta(\phi_t(\bar{z}))$ holds for an appropriate τ. On account of (1) $P_{\bar{z}}(y) \in S_\delta(\bar{z})$, thus $P_z(y) \in S_\delta(z)$ which yields $y \in S_\delta^x(z)$.

(3) Can be proved using the techniques of (1) and (2).

(4) Choose σ such that, if $y \in H_\sigma(x)$, the Poincaré map $P\colon H_\sigma(x) \to H_{\delta_1}(y)$ is defined; where δ_1 is an in lemma 5. Thus, there exists a compact connected set $D \subset S_\delta(y) \cap P(H_\sigma(x))$ joining y to $\partial P(H_\sigma(x))$; thus $P^{-1}(D)$ is the desired set.

(5) Assume that the claim is not true. Then, for arbitrarily small values of σ, $S_\sigma(x)$ separates $H_\alpha(x)$. In particular choose $0 < \sigma < \delta$ as in (3).

Let $\varepsilon > 0$ be such that $f_t(y) = \phi(y, \tau(x, y, t)) \in H_{2\alpha}(\phi_t(x))$, for $y \in H_\alpha(x)$ and $0 \leq t \leq \varepsilon$. Clearly, f_t is an homeomorphism onto its image. Let B be an open set contained in a connected component C of $H_\alpha(x) - S_\sigma(x)$, which does not meet $\partial H_\alpha(x)$.

Then, $f_t(B) \subset f_t(C)$ and $f_t(C)$ is a connected component of $f_t(H_\alpha(x)) - f_t(S_\sigma(x))$, which does not meet $\partial f_t(H_\alpha(x))$.

Clearly, $f_t(S_\sigma(x)) \subset S_\sigma(\phi_t(x))$. Let $z \in f_t(B)$. Any arc a_t joining $\phi_t(x)$ to z and to $\partial f_t(H_\alpha(x))$, must meet a point $p \in S_\sigma(\phi(x,t))$ after z and before $\partial f_t(H_\alpha(x))$. Therefore

$$\text{dist}(\phi_t(x), z) \leq \text{dist}(\phi_t(x), p) \leq \sigma.$$

Decomposing the orbit of x in segments of length ε and reasoning inductively, we have $B \subset S_\sigma(x)$.

Let $z_0 \in B$. By (3), $B \subset S_\delta^x(z_0)$, which implies that z_0 is a stable point of ϕ; a contradiction according to lemma 4. □

Definition. We say that $y \in M$ has local product structure if there exists a homeomorphism of $\mathbb{R}^2$ onto an open neighborhood of y in $H_\alpha(y)$ that maps horizontal (vertical) lines onto open subsets of local stable (unstable) sets in $H_\alpha(y)$.

Lemma 6 shows that Lewowicz's theory applies (see [**7**]) to stable and unstable sets as defined before lemma 6. On account of this we have

Lemma 7. *Except for a finite number of periodic orbits, whose points we call singular, every point of M has local product structure. If x is a singular point, the stable set of x is the union of r arcs, $r \geq 3$, that meet only at x.*

Let M^* be M without the singular points. For $x \in M^*$, let $h_x\colon \mathbb{R}^2 \to H_\alpha(x)$ be such that h_x is some homeomorphism onto its image that gives a local product structure. Define

$$F_x(u,v,t) = \phi(h_x(u,v), \tau(x, h_x(u,v), t))$$

for t in a neighborhood V_x of 0 in such a way that $\{F_x\}_{x \in M^*}$ gives an atlas of M^* that defines a stable foliation W^s whose plaques are $F_x^c = \{F(u,c,t), u \in \mathbb{R}, t \in V_x\}$ and an unstable foliation W^u defined analogously. Let $W^s(x)$ be the leaf of W^s that contains x.

V. Holonomy

In this section we use some concepts of foliations which can be found in [**2**].

For $\sigma < \alpha$ define

$$S(p,\sigma) = \bigcup_{t\in\mathbb{R}^+} S_\sigma(\phi_t(p))$$

$$S(p,\sigma,t_0) = \bigcup_{0\le t\le t_0} S_\sigma(\phi_t(p))$$

Remark. Let $q \in W^s(p)$. Given $\varepsilon > 0$, there exists $T > 0$ such that $\phi_T(q) \in S(p,\varepsilon)$.

To see this, take plaques $\{U_i\}$, $1 \le i \le n$, of $W^s(p)$, and points p_i so that $q \in U_1$, $p \in U_n$ and $p_i \in U_i \cap U_{i+1}$, $1 \le i \le n-1$. Take $\sigma < \varepsilon/n$.

As $\dot{U}$ is negative on C^+ (a set analogous to the set C^- defined in section II) and C^+ is compact, there exists $\overline{T} > 0$ such that if $(p,x) \in C^+$ then $\phi(x,\tau(p,x,t)) \in S(p,\sigma)$ for $t \ge \overline{T}$. Define T_0 as

$$T_0 = \sup\{\tau(p,x,\overline{T}) \mid (p,x) \in C^+\}$$

Then we have $\phi_t(q) \in S(p_1,\sigma)$ and $\phi_t(p_1) \in S(p_2,\sigma)$ for $t \ge T_0$. Next choose $T_1 > 0$ such that $\phi_t(q) \in S(p_2, 2\sigma)$ for $t \ge T_1$. In the same way we find $T > 0$ for which $\phi_T(q) \in S(p, n\sigma)$.

Lemma 8. *Let ξ be a closed curve in $W^s(p)$. Then ξ cannot have one-sided holonomy.* (See [**2**], VII for the definition of one sided holonomy.)

Proof. As in lemma 6 take $0 < \varepsilon < \beta$ and $0 < \sigma < \delta < \alpha$, such that

- if $S_\sigma(x) \cap S_\sigma(y) \neq 0$, then there exists $|t| < \varepsilon$ such that $\bar{y} \in S_\delta(x)$ with $\bar{y} = \phi_t(y)$.
- the connected component, $E_\sigma(x)$, of $S_\delta(x) \cap H_\sigma(x)$ that contains x is the union of arcs joining x to $\partial H_\sigma(x)$.

If the theorem is not true we can distinguish two cases:

(1) There exists a singular point $\bar{p}$ such that $S_\delta(\bar{p}) \cap W^s(p) \neq \phi$.
(2) There does not exist such a singular point.

Suppose first that (2) holds and let ξ be defined by the function $\xi\colon [0,1] \to W^s(p)$, $\xi(0) = \xi(1) = p$.

Assume also that p is not periodic (otherwise the proof is easier). On account of the compactness of $\xi([0,1])$ and the last remark, there exists $m_1 > 0$ so that $\phi(\xi, m_1) \subset S(p,\sigma)$. Denote $\alpha_1 = \phi(\xi, m_1)$ and let t_0 be such that $\alpha_1 \subset S(p,\sigma,t_0)$.

Let $T\colon S(p,\sigma,t_0) \to \mathbb{R}$ be a continuous function for which $\overline{P}(x) \in H_\sigma(\phi(p,t_0))$, where $\overline{P}(x) = \phi(x, T(x))$. Since $\overline{P}(p) = \phi(p,t_0)$ and $\alpha_1 \subset W^s(p)$, $\overline{P}(\alpha_1)$ is contained in the image of an arc $a\colon [-1,1] \to W^s(p)$ with $a(0) = \phi(p,t_0)$.

Set $m_2 = \sup\{T(x) \mid x \in \alpha_1\}$ and $\alpha_2 = \phi(\alpha_1, m_2)$. Then $\alpha_2 \subset A$ where A is given by

$$A = \{\phi(a(s), \tau(a(0), a(s), t)) \mid t \in [0, m_2] \text{ and } s \in [-1,1]\}$$

Since ξ has one-sided holonomy, it is not nullhomotopic. Then α_2 is not nullhomotopic, which implies the existence of numbers $0 \le t_1 \le t_2 \le m_2$ such that

$$S_\sigma(\phi(a(0),t_1)) \cap S_\sigma(\phi(a(0),t_2)) \neq \varnothing$$

because if this were not the case A would be simply connected.

According to this there exists $\bar{y} = \phi(a(0), t + t_2)$, $|t| < \varepsilon$, so that $\bar{y} \in S_\delta(\phi(a(0),t_1))$. Translating $\bar{y}$ and $\phi(a(0),t_1)$ by the flow, we obtain points p_n in the orbit of $a(0)$ and numbers $t_n > \varepsilon$, $\sigma_n \to 0$ such that $\phi(p_n, t_n) \in S_{\sigma_n}(p_n)$.

Now assume that there exists T such that $t_n \le T$. As in lemma 4, choose $T_1 > 0$ such that $\phi(y, \tau(z,y,t)) \in H_{\sigma/2}(\phi_t(z))$, for $y \in E_\sigma(z)$ and $t \ge T_1$.

Take n so that $n\varepsilon > T_1$, and n_0 large such that the distance between whichever two of the first n returnings of the orbit of P_{n_0} to $E_\sigma(p_{n_0})$ is smaller than h, where h is chosen as in lemma 4, i.e. if $\mathrm{dist}(x,y) < h$, then the Poincaré map $P_y\colon H_{\sigma/2}(x) \to H_\sigma(y)$ is defined.

Since $E_\sigma(p_{n_0})$ does not contain singular points, $E_\sigma(p_{n_0})$ is an arc that contains a periodic point $\hat{p}$ of period w.

If t_n were not bounded, a similar argument would work for $t_n > T_1$.

We remark that the orbit of $\hat{p}$ attracts every point of $W^s(p)$.

Take an arc b which is the union of two stable arcs joining $\hat{p}$ to $\partial H_\sigma(\hat{p})$, $b\colon[-1,1]\to W^s(p)$, $b(0)=\hat{p}$, $b([-1,1])\subset \operatorname{clos} H_\sigma(\hat{p})$ and a curve homotopic to ξ contained in the set

$$\{\phi(b(x),\tau(\hat{p},b(s),t)) \ \mid \ t\in[0,w] \text{ and } s\in[-1,1]\}$$

which can be contracted to the orbit of $\hat{p}$. Then, ξ is homotopic to a multiple of that orbit.

Now assume that (1) holds. Let $\bar{p}$ be a singular point of period v. Let $0<\bar{\sigma}<\bar{\delta}$ be, as in the previous case, such that $E_{\bar{\sigma}}(\bar{p})$ is the union of r arcs b_k, $1\le k\le r$, $b_k\colon[0,1]\to H_{\bar{\sigma}}(\bar{p})$, $b_k(0)=\bar{p}$ in such a way that the first return map $P_1\colon E_{\bar{\sigma}}(\bar{p})\to S_\delta(\bar{p})$ is well defined.

Let n_k be the less integer such that $P_1^{n_k}(b_k)\cap b_k\neq\{\bar{p}\}$.

Now we can find, as above, a number k and a curve homotopic to ξ contained in the set B defined as

$$B=\{\phi(b_k(s),\tau(\bar{p},b_k(s),t)) \ \mid \ t\in[0,n_kv] \text{ and } s\in[0,1]\}$$

Observe that B is a cylinder. Thus there is an integer j such that for every $\varepsilon_0>0$, ξ is homotopic to $\beta\subset B$, where β is defined as follows:

Let $\eta\colon[0,jn_kv]\to B$ be defined by $\eta(t)=\phi(\eta(0),\tau(\bar{p},\eta(0),t))$ where $\eta(0)\in b_k\cap H_{\varepsilon_0}(\bar{p})$. Let γ be an arc contained in b_k joining $\eta(jn_kv)$ to $\eta(0)$. Finally set $\beta=\eta*\gamma$.

Since ξ has one-sided holonomy, β also has.

Take arcs I, $I\subset U_\delta(\eta(0))$ and J, $J\subset U_\delta(\eta(jn_kv))$ so that the holonomy map of γ, $g\colon J\to I$ and the Poincaré map of ϕ, $Q\colon I\to J$, are defined.

Then $g\circ Q$ is the holonomy map of β. Let I_1 be the component of $I-\{\eta(0)\}$ such that $g\circ Q|I_1=id$.

As the number of leaves of W^s that satisfy (1) is finite, the set of plaques of those leaves that meet I and J is countable. Therefore we can find, arbitrarily close to $\eta(0)$, a point $x_0\in M^*\cap I_1$.

Now consider the curve $\widetilde{\beta}=\widetilde{\eta}*\widetilde{\gamma}$ where $\widetilde{\eta}$ is an arc of ϕ-trajectory joining x_0 to $g^{-1}(x_0)$, and $\widetilde{\gamma}$ is a stable arc joining $g^{-1}(x_0)$ to x_0. (Observe that the holonomy of $\widetilde{\beta}$ is trivial in a transversal that contains

x_0.)

If we choose ε_0 small, the arguments of the previous case prove that $\widetilde{\beta}$ is homotopic to a multiple of a periodic orbit in M^*.

In whichever of the considered cases we find a periodic point, $\hat{p}$, such that the holonomy of a multiple of the orbit of $\hat{p}$ is trivial in an unstable arc, which is absurd. □

Lemma 9. *There does not exist a closed nullhomotopic transversal to W^s.*

Proof. Assume, on the contrary, that γ is nullhomotopic and transversal to W^s. Then there exists a continuous map $A: D^2 \to M$ so that $A|_{\partial D^2} = \gamma$.

A can be put in general position with respect to W^s by arguments in [**2**] (VII) and in general position with respect to the singular orbits.

Then $A^*(W^s)$ is a foliation with singularities $x_1, \ldots, x_n, y_1, \ldots, y_m$ such that the points $A(x_i)$ belong to the singular orbits, the points y_i are orientable saddles or centres, and there is no saddle connection between distinct saddles.

A can be chosen such that there is no connection between a saddle and a point x_i.

We can also modify A so that there is no connection between two of the points x_i, because if x_k and x_l were connected, $A(x_k)$ and $A(x_l)$ would belong to the same singular orbit, by arguments of the previous section. Then we can modify A to eliminate x_k and x_l.

Let C stand for the set of cycles of $A^*(W^s)$ (see [**2**], VII, 2.1).

Let K_σ be the connected component in C of a centre σ. Arguments of ([**2**], VII) show that ∂K_σ is a cycle. If ∂K_σ is a circle leaf it is easy to check that it has one sided holonomy.

Then assume that for every centre $\sigma, \partial K_\sigma$ contains a saddle. We claim that there exists a centre σ_0 so that the saddle in ∂K_{σ_0} is completely self-connected, because if this is not the case we get a contradiction according to the Poincaré-Hopf index formula for $A^*(W^s)$.

Now it is possible to order by inclusion the so obtained saddle self-connections. Since there is only a finite number of them we can find one, say β, which is one sided. Thus β induces one-sided holonomy in W^s; a contradiction according to lemma 8. □

VI. Exponential growth

(In this section r is as in lemma 7). Let $0 < \sigma < \delta$ be such that the connected component of $U_\delta(x) \cap H_\sigma(x)$ that contains x is the union of arcs, $U_\sigma^i(x)$, $1 \geq i \leq r$, joining x to $\partial H_\sigma(x)$.

Let $D_i(x)$, $1 \leq i \leq r$, be discs, with $\operatorname{diam}(D_i(x)) \leq \sigma$, contained in plaques of W^s such that $D_i(x) \cap U_\sigma^i(x) \neq \varnothing$ and the endpoints of $U_\sigma^i(x)$ do not belong to $D_i(x)$.

For y close to x and $1 \leq i \leq 2$, there exist $1 \leq q_i \leq r$ such that $U_\sigma^i(y) \cap D_{q_i}(x) \neq \varnothing$ and the endpoints of $U_\sigma^i(y)$ do not belong to $D_{q_i}(x)$.

Now, since M is compact, there exist $\varepsilon > 0$ and points x_i, $1 \leq i \leq K$, so that if $y \in B_\varepsilon(x_j)$ and $1 \leq i \leq 2$, there exist $1 \leq q_i \leq r$ so that $U_\sigma^i(y) \cap D_{q_i}(x_j) \neq \varnothing$ and the endpoints of $U_\sigma^i(y)$ do not belong to $D_{q_i}(x_j)$.

Choose $0 < \rho < \sigma/2$ such that if $\operatorname{dist}(x, y) \leq \rho$, then

$$\operatorname{dist}(\phi(y, \tau(x, y, 0)), x) \leq \sigma/2.$$

Take $t^* > 0$ so that for every $p \in M$ and $x, y \in U_\delta(p)$ with $\operatorname{dist}(x, y) \geq \rho$ we have

$$\operatorname{dist}(\phi(y, \tau(x, y, t)), \phi(x, t)) \geq \sigma$$

for some t, $0 \leq t \leq t^*$.

Define T as

$$T = \sup\{\tau(x, y, t) \ \mid \ x, y \in U_\delta(p), \operatorname{dist}(x, y) \geq \rho, \\ \operatorname{dist}(\phi(x, t), \phi(y, \tau(x, y, t))) \leq \sigma \text{ and } 0 \leq t \leq t^*\}.$$

Let us define the ε-length of a curve. Assume that ε is a small positive number and $\gamma: [a, b] \to M$ is a continuous curve. The ε-length of γ, which is denoted by $L_\varepsilon(\gamma)$, is the maximum number n of points $a \leq t_1 < t_2 < \ldots < t_n \leq b$ such that $\operatorname{dist}(\gamma(t_i), \gamma(t_{i+1})) = \varepsilon$.

For $q \in M$ take the curve $\xi_q: [0, T] \to M$ given by $\xi_q(t) = \phi(q, t)$. Define A as

$$A = \sup\{L_\sigma(\xi_q) \ \mid \ q \in M\}$$

Let $x_0 \in M^*$ be a point such that $U_\delta(x_0)$ contains no periodic points of ϕ.

Now define a family of curves α_n in the following way: α_1 is an arc, $\alpha_1\colon [0,1] \to U_\delta(x_0)$, $\alpha_1([0,1]) \subset \operatorname{clos} H_\sigma(x_0)$, $\alpha_1(0) = x_0$, $\alpha_1([0,1]) \cap \partial H_\sigma(x_0) \neq \varnothing$.

Assume that α_n is defined as

$$\alpha_n = \alpha_n^1 * \alpha_n^{1,2} * \alpha_n^2 * \alpha_n^{2,3} * \cdots * \alpha_n^{2^n}$$

where $\alpha_n^k\colon [0,1] \to U_\delta(\alpha_n^k(0))$ is an arc such that

$$\alpha_n^k([0,1]) \subset \operatorname{clos} H_\sigma(\alpha_n^k(0)) \text{ and } \alpha_n^k([0,1]) \cap \partial H_\sigma(\alpha_n^k(0)) \neq \varnothing$$

and $\alpha_n^{k,k+1}$ is an arc of ϕ-trajectory, joining $\alpha_n^k(1)$ to $\alpha_n^{k+1}(0)$ with σ-length $\leq An$.

Next we define α_{n+1}. Because of the way t^* was chosen, there exist numbers $0 \leq t_k \leq t^*$, $0 \leq \hat{t}_k \leq t^*$ and $0 \leq s_k \leq 1$ such that

$$g_{n+1}^{2k}(s) \stackrel{\text{def}}{=} \phi(\alpha_n^k(s), \tau(\alpha_n^k(s_k), \alpha_n^k(s), t_k)) \in \operatorname{clos} H_\sigma(\phi(\alpha_n^k(s_k), t_k))$$

for $s \in [s_k, 1]$ and

$$g_{n+1}^{2k-1}(s) \stackrel{\text{def}}{=} \phi(\alpha_n^k(s), \tau(\alpha_n^k(0), \alpha_n^k(s), \hat{t}_k)) \in \operatorname{clos} H_\sigma(\phi(\alpha_n^k(0), \hat{t}_k))$$

for $s \in [0, s_k]$. Then we have that

$$g_{n+1}^{2k}([s_k, 1]) \cap \partial H_\sigma(\phi(\alpha_n^k(s_k), t_k)) \neq \varnothing$$

and

$$g_{n+1}^{2k-1}([0, s_k]) \cap \partial H_\sigma(\phi(\alpha_n^k(0), \hat{t}_k)) \neq \varnothing$$

Define

$$\alpha_{n+1}^{2k}(s) = g_{n+1}^{2k}(s + (1-s)s_k) \text{ and } \alpha_{n+1}^{2k-1}(s) = g_{n+1}^{2k-1}(ss_k)$$

Finally, define $\alpha_{n+1}^{2k-1,2k}$ as the arc of ϕ trajectory joining $\alpha_{n+1}^{2k-1}(1)$ to $\alpha_{n+1}^{2k}(0)$.

Remark. The σ-length of $\alpha_{n+1}^{k,k+1}$ is $\leq A(n+1)$.

Then according to the last paragraph, given n, there exist j and m such that for at least $2^n/Kr$ distinct values of i, we have $\alpha_n^i \cap D_m(x_j) \neq \varnothing$ and the endpoints of α_n^i do not belong to $D_m(x_j)$.

Let N be the maximum integer of $2^n/Kr$ and let $0 < t_1 < t_2 < \ldots < t_N < 1$ be numbers so that $\alpha_n(t_i) \in D_m(x_j)$.

For $1 \leq p \leq N$, define $\eta_p(s) = \alpha_n(s)$ for $s \in [t_1, t_p]$.

Let β_p be an arc joining $\alpha_n(t_p)$ to $\alpha_n(t_1)$, whose image is contained in $D_m(x_j)$ and set $\gamma_p = \eta_p * \beta_p$

Lemma 10. $\gamma_p \cong \gamma_q$ *if and only if* $p = q$

Proof. Assume that $\gamma_p \cong \gamma_q$ with $p > q$. Then $\beta_q^{-1} * \eta_q^{-1} * \eta_p * \beta_p \cong 0$. Observe that $\eta_q^{-1} * \eta_p$ is homotopic (with endpoints fixed) to the curve η defined by

$$\eta = \alpha_n|_{[t_q, t_p]}.$$

Let us show the existence of a curve $\widehat{\eta}$ transversal to W^s and homotopic (with endpoints fixed) to η. Choose numbers $t_q = a_1 < b_1 < a_2 < \ldots a_l < b_l = t_p$ such that $\eta|_{[a_i, b_i]}$ is transversal to W^s and $\eta|_{[b_i, a_{i+1}]}$ is an arc of ϕ-trajectory.

A lemma of trivialization (see [**2**]) shows that there exist neighborhoods V_i of $\eta([b_i, a_{i+1}])$ in which the foliation W^s is trivial. Now it is easy to modify η inside V_i to obtain $\widehat{\eta}$. Our assumption implies $\beta_q^{-1} * \widehat{\eta} * \beta_p \cong 0$, which is homotopic to a closed transversal to W^s. But this cannot be true, according to lemma 9. □

Lemma 11. γ_p *is homotopic to a curve* $\widetilde{\gamma}_p$ *of* σ*-length* $\leq 3 + 2An$ *for* $1 < p \leq N$.

Proof. α_n was constructed so that there exists a continuous function $\tau: [0,1] \to \mathbb{R}$ such that $\tau(s) \leq nT$ and

$$\phi(\alpha_n(s), -\tau(s)) \in H_\delta(x_0)$$

for $s \in [0,1]$. Define $B: [0,1] \times [0,1] \to M$ by

$$B(s,t) = \phi(\alpha_n(s), -t\tau(s))$$

Let λ_p be a curve homotopic (with endpoints fixed) to $B(.,1)$ with length $\leq 2\sigma$ and set

$$G_s(t) = B(t_1(1-s) + st_p, t)$$
$$\widetilde{\eta}_p = G_0 * \lambda_p * G_1^{-1} \cong \eta_p.$$

Then $\widetilde{\gamma}_p = \widetilde{\eta}_p * \beta_p$ satisfies the lemma. □

Corollary. *$P(3\sigma + 2A\sigma n) \geq N$, where P is the growth function defined in the Introduction. Observe that this inequality gives the desired theorem.*

Note. After this work was completed I was informed that Inaba and Matsumoto ([4]) obtained another non existence theorem for expansive flows on certain 3-manifold.

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